Nonlinear Acoustics: Fundamental Concepts and Shock Applications[1]

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Outline of Lecture

- **PART I: Introduction**
- **PART II: Lossless Propagation Under Euler's System**
- **PART III: Propagation in Thermoviscous Gases**
- **PART IV: Poroacoustic Propagation Under Darcy's Law**
- **PART V: Closure**

PART I: Terminology, Hierarchy Chart, History, & Concepts

Terminology

1. **Compressible flow:** Fluid flow in which $\rho(> 0)$, the mass density, *cannot* be treated as constant.
2. **Acoustics:** Irrotational compressible flow.
3. **Irrotational flow:** One in which $\nabla \times \mathbf{v} = (0, 0, 0)$ everywhere; here, $\mathbf{v} = (u, v, w)$ is the velocity vector.
4. **Longitudinal waves:** Wave motion in which the particle displacement is *parallel* to the direction of propagation.
5. **Isentropic flow:** Flows wherein $D\eta/Dt = 0$; here, $\eta :=$ specific entropy and $D/Dt :=$ material derivative.
6. **Homentropic flow:** The special case of isentropic flow in which η, the specific entropy, is *constant*.
7. **Adiabatic process:** One that occurs without the transfer of heat between a system and its surroundings.

Terminology (Cont.)

8. **Homogeneous medium:** One that, absent perturbation, exhibits uniform properties throughout.

9. **Mach number:**

$$\mathrm{Ma} = \frac{|\text{(Maximum source velocity)}|}{c_0},$$

   where $c_0 :=$ speed of sound in undisturbed fluid.

10. **Shock wave:** Propagating surface across which the field variables (e.g., density) suffer a jump discontinuity.

11. **Traveling Wave Solution (TWS):** Of a (1D) nonlinear PDE is one with argument $x - Vt$; here, $V :=$ const. w/ units m/s.

12. **Dispersed shock:** Non-increasing TWS that approaches constant, but unequal, limits at $x = \pm\infty$.

Within the Grand Scheme . . .

Continuum Physics (i.e., “Field Theories”)

Fluid Mechanics

Fluid Dynamics

Compressible Flow

Irrotational Compressible Flow (a.k.a. “Acoustics”)

Finite-amplitude Acoustics (“small but finite” Ma, i.e., Ma $\ll 1$)

Linear Acoustics (“infinitesimal” Ma, i.e., Ma $\to 0$)

Nonlinear Acoustics: Some Historical Highlights

1755 Euler presents his (incomplete) system of equations for lossless compressible flow; *realistic* equation of state (EoS) lacking.

1816 Laplace: Sound propagation is *adiabatic*, not isothermal

1848 Stokes discovers "surface of discontinuity"

1860 Earnshaw: Exact 1D EoM for homentropic propagation in gases

1870 Rankine: "Wave of permanent form" in heat conducting gases

1910 Rayleigh: Solves 1D "viscosity-only" Navier–Stokes sys.

1922 Becker: Exact solution to 1D Navier–Stokes–Fourier (NSF) sys.

1927 Madelung: Link between Schrödinger's Eq. and the Euler sys.

1949 Lagerstrom *et al.* apply "Cole–Hopf transform" to Burgers' Eq.

1956 Lighthill: Burgers' Eq. with "diffusivity of sound" coefficient

1963 Blackstock: Finite-amplitude NSF sys. for compressible flow

1971 Kuznetsov: Weakly-nonlinear, 3rd order, EoM for finite-amplitude propagation in thermoviscous gases

Historical Highlights (Cont.)

- 1979 Crighton: Survey of the equations of nonlinear acoustics
- 1982 Mobbs: Notes the most general Lagrangian density for Euler sys.
- 1995 Green & Naghdi: Thermal displacement-based theory of compressible flow
- 2004 Jordan obtains exact TWS of Kuznetsov's acoustic Eq.
- 2010 Straughan formulates “Cattaneo–Christov” theory of lossless compressible flow
- 2012 Brenner: Bi-velocity generalization of acoustic equations
- 2014 Brunnhuber & Kaltenbacher: New, single-equation, special case of Blackstock's (1963) compressible flow sys.
- 2015 Christov & Jordan corrects variational-based acoustic EoM given by Morse & Ingard (1968)
- 2017 Nikolić & Kaltenbacher: Shape optimization analysis of acoustic lens based on the weakly-nonlinear Westervelt Eq.
- 2020 Scholle: Derives new, 5th order, weakly-nonlinear, EoM based on a variational approach

Concepts, Notation, etc.

1. **Perfect gas:** One that obeys the law (Thompson, 1972)

$$\wp = (c_\mathrm{p} - c_\mathrm{v})\rho\vartheta \qquad (c_\mathrm{p}, c_\mathrm{v} := \text{constant}),$$

   where: $\wp :=$ thermodynamic pressure, $\vartheta :=$ absolute temperature, and $c_\mathrm{p} > c_\mathrm{v} > 0$ are the specific heats.

2. **Ratio of specific heats:** We let $\gamma = c_\mathrm{p}/c_\mathrm{v}$, where $1 < \gamma \leq 5/3$ for perfect gases (Note: $\gamma = 1.4$ for air; $\gamma = 1.67$ for Ar and He).
3. **Subscript "0":** The const., equilibrium value of the quantity.
4. **The Gibbs relation (perfect gases):** $\vartheta \mathrm{d}\eta = c_\mathrm{v}\mathrm{d}\vartheta - (\wp/\rho^2)\mathrm{d}\rho$, defines the specific entropy η (Thompson, 1972).
5. **Conserved quantities:** Mass, Momentum, (total) Energy.
6. **Conservation law:** $\mathcal{E}_t + \boldsymbol{\nabla} \cdot \mathbf{Q} = 0$.
   Here, $\mathcal{E} :=$ (conserved quantity)/(volume), $\mathbf{Q} :=$ flux law.
7. **Balance law:** $\mathcal{E}_t + \boldsymbol{\nabla} \cdot \mathbf{Q} = S$. Here, $S :=$ source term.
8. **Laplacian (scalar):** $\nabla^2\varphi = \boldsymbol{\nabla} \cdot (\nabla\varphi)$.

PART II: Lossless Propagation Under Euler's System

Review of Linear Acoustics

For the case of a homogeneous fluid experiencing homentropic flow, the linearized version of Euler's system of equations takes the form (Lamb, 1945; Pierce, 1989):

Conservation of Mass:

$$\partial\rho/\partial t + \rho_0(\boldsymbol{\nabla}\cdot\mathbf{v}) = 0, \tag{1}$$

Momentum Conservation Equation:

$$\partial(\rho_0\mathbf{v})/\partial t = \rho_0\boldsymbol{f} - \nabla p \qquad (\boldsymbol{f} := \text{body force vector}), \tag{2}$$

Equation of State (EoS):

$$p = \wp - \wp_0 = c_0^2(\rho - \rho_0) \quad \text{(homentropic flow)}. \tag{3}$$

$c_0 = \sqrt{\gamma\wp_0/\rho_0}$ is the "small amplitude" sound speed in the fluid. A subscript "0" denotes the (const.) equilibrium state value.

Linear Acoustics – The Wave Equation

Eliminating ρ between the conservation of mass Eq. and the EoS, and then applying $\partial/\partial t$ to the result, yields

$$\partial^2 p/\partial t^2 = -\rho_0 c_0^2 \partial(\boldsymbol{\nabla} \cdot \mathbf{v})/\partial t. \tag{4}$$

Now applying $\boldsymbol{\nabla}\cdot$ to both sides of the momentum Eq. yields

$$\rho_0 \boldsymbol{\nabla} \cdot (\partial \mathbf{v}/\partial t) = -\nabla^2 p \qquad [\boldsymbol{f} = (0,0,0)]. \tag{5}$$

Finally, eliminating $\boldsymbol{\nabla} \cdot (\partial \mathbf{v}/\partial t)$ between the former and latter equations yields, after simplifying, the well known *wave equation*

$$\partial^2 p/\partial t^2 = c_0^2 \nabla^2 p \qquad (p = \wp - \wp_0). \tag{6}$$

Remark: Compare Eq. (6) with EoM for 1D string (d'Alembert, 1747)

$$\frac{\partial^2 Y}{\partial t^2} = \mathfrak{a}_0^2 \frac{\partial^2 Y}{\partial x^2}. \qquad (\mathfrak{a}_0 := \text{speed of transverse waves}). \tag{S}$$

Transverse displacement $Y = Y(x,t)$ relative to x-axis.

Euler's Equations of Compressible Flow

In the case of a homogeneous fluid experiencing homentropic flow, Euler's system of equations can be expressed as

Conservation of Mass:

$$\rho_t + \boldsymbol{\nabla} \cdot (\rho \mathbf{v}) = 0. \qquad (I)$$

Momentum Equation:

$$\rho \left[\mathbf{v}_t + \tfrac{1}{2}\nabla |\mathbf{v}|^2 - \mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{v}) \right] = -\nabla \wp + \rho \boldsymbol{f}. \qquad (II)$$

Equation of State (EoS):

$$\wp = \Pi(\rho). \qquad (III)$$

Entropy Equation:

$$\eta = \eta_0 \qquad (IV)$$

Here, $\boldsymbol{f}$ is the per unit mass body force vector.

Some Thermodynamics: Equations of State

General EoS for *perfect gases*, obtained via the Gibbs relation:

$$\wp = \wp_0(\rho/\rho_0)^{\gamma} \exp\left[(\eta - \eta_0)/c_{\mathrm{v}}\right]. \tag{7}$$

If $|\eta - \eta_0| \ll |\rho - \rho_0|$ are "small but finite", then Eq. (7) yields

$$\wp - \wp_0 \approx \rho_0 c_0^2 \left[\frac{\rho - \rho_0}{\rho_0} + \frac{\gamma - 1}{2}\left(\frac{\rho - \rho_0}{\rho_0}\right)^2 + (\eta - \eta_0)/c_{\mathrm{p}}\right]. \tag{8}$$

If the flow is *homentropic* (i.e., $\eta = \eta_0$), then Eqs. (7) and (8) become

$$\wp = \wp_0(\rho/\rho_0)^{\gamma}, \tag{9}$$

$$\wp - \wp_0 \approx \rho_0 c_0^2 \left[\frac{\rho - \rho_0}{\rho_0} + \frac{\gamma - 1}{2}\left(\frac{\rho - \rho_0}{\rho_0}\right)^2\right]. \tag{10}$$

Recall: $c_0 = \sqrt{\gamma\wp_0/\rho_0}$ and $1 < \gamma \leq 5/3$.

Note that *linearizing* Eq. (10) yields the well known result:

$$p := \wp - \wp_0 \approx c_0^2(\rho - \rho_0) \qquad \text{(linearized, homentropic flow)}.$$

Single, Exact, EoM: Critical Assumptions

1. Homogeneous-quiescent (fluid) medium of propagation.
2. Fluid exhibits perfect gas behavior (e.g., air).
3. Homentropic flow.
4. $\boldsymbol{f} = (0, 0, 0)$.
5. $\boldsymbol{\nabla} \times \mathbf{v} = (0, 0, 0)$ everywhere at time $t = 0$.

Assumption (1) implies that, in its ambient state, the fluid is motionless and has uniform properties.
Together, Assumptions (2) & (3) imply, among other things, that

$$\wp = \wp_0(\rho/\rho_0)^\gamma \qquad (1 < \gamma \leq 5/3).$$

Assumption (4) implies that gravity is neglected.
Assumption (5) implies $\mathbf{v} = \nabla\phi$, where $\phi = \phi(x, y, z, t)$ is the *scalar* velocity potential, and *only* longitudinal waves occur.

Exact Equation For Propagation in Gases

Under the just-invoked assumptions, the Euler system can be reduced to a *single* EoM, viz. (Thompson, 1972):

$$c^2\nabla^2\phi - \phi_{tt} - \partial_t|\nabla\phi|^2 - \tfrac{1}{2}(\nabla\phi)\cdot\nabla|\nabla\phi|^2 = 0, \qquad (11)$$

where the square of the *instantaneous* sound speed is given by

$$c^2 = c_0^2 - (\gamma - 1)\left(\phi_t + \tfrac{1}{2}|\nabla\phi|^2\right), \qquad (12)$$

and $\wp$ can be expressed as (Morse & Ingard, 1968)

$$\wp = \wp_0\left[1 - c_0^{-2}(\gamma - 1)\left(\phi_t + \tfrac{1}{2}|\nabla\phi|^2\right)\right]^{\frac{\gamma}{\gamma-1}}. \qquad (13)$$

Recall:
$\phi = \phi(x, y, z, t)$ is the *scalar* velocity potential,
$c_0 = \sqrt{\gamma\wp_0/\rho_0}$ is the small-amplitude sound speed in the gas,
A subscript “0" denotes the quantity’s, const., equilibrium-state value.

Eqs. of Finite-Amplitude Lossless Acoustics

Under the finite-amplitude scheme, i.e., $\mathrm{Ma} \ll 1$ and $|s| \sim \mathcal{O}(\mathrm{Ma})$, where $s = (\rho - \rho_0)/\rho_0$, our exact EoM [i.e., Eq. (11)] yields, on neglecting $\mathcal{O}(\mathrm{Ma}^2)$ terms, the weakly-nonlinear acoustic model

$$c_0^2[1 - c_0^{-2}(\gamma - 1)\phi_t]\nabla^2\phi - \phi_{tt} = \partial_t|\nabla\phi|^2 \quad \text{(Blackstock, 1963).} \qquad (14)$$

Over the years, several variants of Eq. (14) have been derived; viz.:

i. Lossless Kuznetsov Eq. (Kuznetsov, 1971)

$$c_0^2\nabla^2\phi - \phi_{tt} = \partial_t[|\nabla\phi|^2 + c_0^{-2}(\beta - 1)(\phi_t)^2]$$

ii. Lossless Lighthill–Westervelt Eq. (Coulouvrat, 1992)

$$c_0^2\nabla^2\phi - \phi_{tt} = c_0^{-2}\beta\partial_t[(\phi_t)^2]$$

iii. Lossless RSGC Eq. (Rasmussen *et al.*, 2008)

$$(c_0^2 - \phi_t)\nabla^2\phi - \phi_{tt} = \partial_t[|\nabla\phi|^2 + c_0^{-2}(\beta - \tfrac{3}{2})(\phi_t)^2]$$

$\beta = \frac{1}{2}(\gamma + 1)$ is the *coefficient of nonlinearity.*

Equations of Linear Acoustics Re-derived

Linearizing, i.e., neglecting all powers, products, etc., of ϕ (+ its derivatives), under the assumption that acoustic disturbances exhibit **very small** (i.e., $\mathrm{Ma} \to 0$) amplitudes, reduces our exact, nonlinear EoM to a *linear* EoM; specifically,

$$c_0^2 \left[1 - c_0^{-2}(\gamma - 1)\left(\phi_t + \tfrac{1}{2}|\nabla\phi|^2\right)\right]\nabla^2\phi - \phi_{tt} - \partial_t|\nabla\phi|^2 - \tfrac{1}{2}(\nabla\phi)\cdot\nabla|\nabla\phi|^2 = 0,$$

becomes, on neglecting the red terms, the simple wave Eq.

$$c_0^2\nabla^2\phi - \phi_{tt} = 0 \qquad (\text{Recall: } \mathbf{v} = \nabla\phi). \tag{15}$$

Eq. (15) can be expressed in terms of either the acoustic pressure (i.e., p) or the *condensation* (i.e., s) using the (linearized) relations

$$p = \wp - \wp_0 = -\rho_0\phi_t \qquad \text{and} \qquad s = \frac{\rho - \rho_0}{\rho_0} = -c_0^{-2}\phi_t. \tag{16}$$

PART III: Propagation in Thermoviscous Gases

Generalization to Thermoviscous Gases: The NSF Sys.

In 1D, the mass, momentum, and entropy equations of the Navier–Stokes–Fourier (NSF) sys. become

$$\rho_t + u\rho_x + \rho u_x = 0, \tag{17}$$

$$\rho(u_t + uu_x) = -\wp_x + (\tfrac{4}{3}\mu + \mu_{\rm b})u_{xx}, \tag{18}$$

$$\rho\theta(\eta_t + u\eta_x) = K\vartheta_{xx} + (\tfrac{4}{3}\mu + \mu_{\rm b})(u_x)^2, \tag{19}$$

where this sys. is closed with the addition of the EoS

$$\wp = \wp_0(\rho/\rho_0)^\gamma \exp\left[(\eta - \eta_0)/c_{\rm v}\right]. \tag{20}$$

Remark: This is the *exact* (1D) NSF sys. for thermoviscous perfect gases with constant transport coefficients.

The velocity has the form $\mathbf{v} = (u(x,t), 0, 0)$, where $\nabla \times \mathbf{v} = \mathbf{0}$, $\mu(> 0)$, $\mu_{\rm b}(\geq 0)$ are the (constant) shear and bulk viscosities, $K(> 0)$ is the (constant) thermal conductivity.

Finite-Amplitude Thermoviscous Acoustic Models

If $\mathrm{Ma} \ll 1$, i.e., under the finite-amplitude scheme, neglecting terms $\mathcal{O}(\mathrm{Ma}^2)$ allows us to reduce the exact NSF sys. to the approx. EoM:

BLSC Eq. (Blackstock, 1963; Lesser & Seebass, 1968; Crighton, 1979)

$$c_0^2[1 - c_0^{-2}(\gamma - 1)\phi_t]\phi_{xx} - \phi_{tt} + \delta\phi_{txx} = \partial_t(\phi_x)^2,$$

the (bi-directional) variants of which are:

Kuznetsov Eq. (Kuznetsov, 1971)

$$c_0^2\phi_{xx} - \phi_{tt} + \delta\phi_{txx} = \partial_t[(\phi_x)^2 + c_0^{-2}(\beta - 1)(\phi_t)^2]$$

Lighthill–Westervelt Eq. (Coulouvrat, 1992)

$$c_0^2\phi_{xx} - \phi_{tt} + \delta\phi_{txx} = c_0^{-2}\beta\partial_t[(\phi_t)^2]$$

RSGC Eq. (Rasmussen *et al.*, 2008)

$$(c_0^2 - \phi_t)\phi_{xx} - \phi_{tt} + \delta\phi_{txx} = \partial_t[(\phi_x)^2 + c_0^{-2}(\beta - \tfrac{3}{2})(\phi_t)^2]$$

$\delta \propto \mu$ is the *diffusivity of sound* – effects of *internal* loss mechanisms.

Thermoviscous TWS: Kuznetsov's Eq.

In dimensionless form, the 1D Kuznetsov equation reads

$$\Phi_{xx} - \Phi_{tt} + (\mathrm{Re_d})^{-1}\Phi_{txx} = \mathrm{Ma}\,\partial_t\{(\Phi_x)^2 + (\beta-1)(\Phi_t)^2\} \quad (\mathrm{Ma} \ll 1),$$

where Φ is the *dimensionless* form of ϕ and $\mathrm{Re_d} \propto 1/\delta$ is a Reynolds number. It admits a *Dispersed shock* TWS:

$$\Phi_x(x,t) = \tfrac{1}{2}\{1 - \tanh[2(x - Vt)/L]\} \qquad (V > 1), \qquad (21)$$

where $V = V(\beta, \mathrm{Ma})$ and L is the "shock thickness":

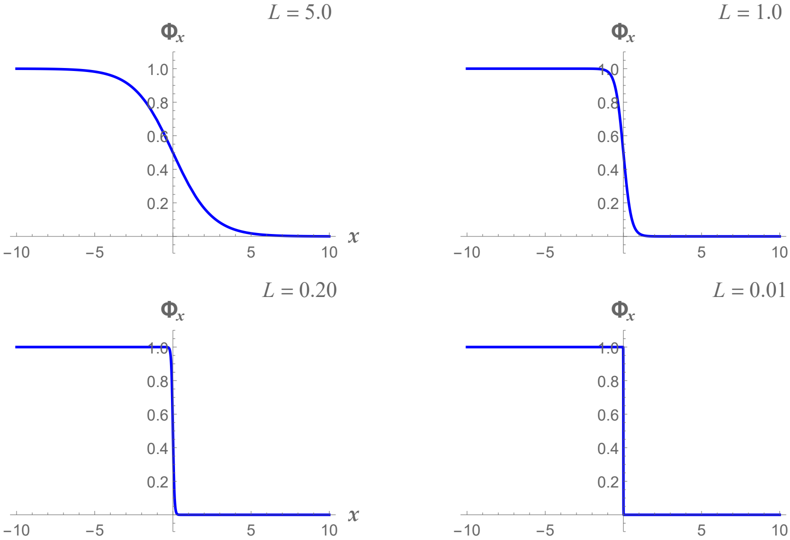
$$L \propto 1/\mathrm{Re_d}.$$

Physically, this TWS describes the velocity of a gas in the context of the *Piston problem*, on the real-line $x \in \mathbb{R}$, with the piston at $x = -\infty$ and moving to the right (i.e., *compressing* the gas) with constant speed $= 1 < V$.

NOTE: The shock speed V suffers a bifurcation at $\mathrm{Ma} = \mathrm{Ma_c}$, where $\mathrm{Ma_c}$ is a critical Mach number value (Jordan, 2004).

Steepening of Kuznetsov TWS as $L \to 0$

Φ_x vs. x for air. Blue curves: “tanh” TWS of Kuznetsov’s Eq.



PART IV: Poroacoustic Propagation Under Darcy's Law

Poroacoustics Based on Darcy's Drag Law

In the case of a rigid-solid porous matrix – one which is homogeneous and isotropic – that is permeated by a perfect gas, finite-amplitude acoustic propagation in the gas can be modeled by the following sys. based on *Darcy's drag law:*

$$\rho_t + \boldsymbol{\nabla} \cdot (\rho \mathbf{v}) = 0, \tag{22}$$

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \frac{1}{2} \nabla |\mathbf{v}|^2 \right) = -\nabla \wp - \left(\frac{\mu \chi}{\mathcal{K}} \right) \mathbf{v} \qquad (\boldsymbol{\nabla} \times \mathbf{v} = \mathbf{0}), \tag{23}$$

$$\eta \approx \eta_0 \qquad \text{(homentropic approx.)}, \tag{24}$$

$$\wp - \wp_0 \approx \rho_0 c_0^2 \left[\frac{\rho - \rho_0}{\rho_0} + \frac{1}{2}(\gamma - 1) \left(\frac{\rho - \rho_0}{\rho_0} \right)^2 \right], \tag{25}$$

$$c_0 = \sqrt{\gamma \wp_0 / \rho_0}. \tag{26}$$

Here,
$\mathcal{K} > 0$ and $\chi \in (0, 1)$ are the (const.) permeability and porosity.

Darcy-Based, Finite-Amplitude EoM

Under the finite-amplitude approximation, our Darcy-based system yields a single poroacoustic EoM, which in "Kuznetsov form" reads (Jordan, 2005):

$$\nabla^2\Phi - \frac{\partial^2\Phi}{\partial t^2} - \sigma\frac{\partial\Phi}{\partial t} = \mathrm{Ma}\frac{\partial}{\partial t}\Big[\,|\nabla\Phi|^2 + (\beta - 1)\Big(\frac{\partial\Phi}{\partial t}\Big)^2\Big], \quad (27)$$

and which is stated here in terms of *dimensionless* variables.

Note:
Φ is the (dimensionless) *scalar velocity potential*,

Again, $0 < \mathrm{Ma} \ll 1$ is the *Mach number*,

$\sigma :=$ dimensionless *Darcy coefficient*, where $\sigma \sim \mathcal{O}(\mathrm{Ma})$.

Recall: For perfect gases, $\beta = \frac{1}{2}(\gamma + 1)$.

Signaling Problem: Sinusoidal Input at Surface of Porous Slab

In this 1D problem, our (dimensionless) poroacoustic EoM is written in "Lighthill–Westervelt form". For a *compressive* signal of frequency $\omega > \alpha^\star$, we have (Jordan, 2005):

$$\begin{cases} p_{xx} - (1 - 2\beta \mathrm{Ma} p) p_{tt} + 2\beta \mathrm{Ma} (p_t)^2 = \sigma p_t, & (x,t) \in (0,1) \times (-\infty, \dagger), \\ p(0,t) = H(t) \sin(\omega t), \quad p(1,t) = 0, \quad t \in (-\infty, \dagger), \\ p(x,0) = 0, \quad p_t(x,0) = 0, \quad x \in (0,1), \qquad \text{where} \qquad \dagger = \min(1, t_{\mathrm{sh}}). \end{cases}$$

Dimensionless acoustic pressure: $p = -\Phi_t$,

Heaviside unit step function: $H(\cdot)$,

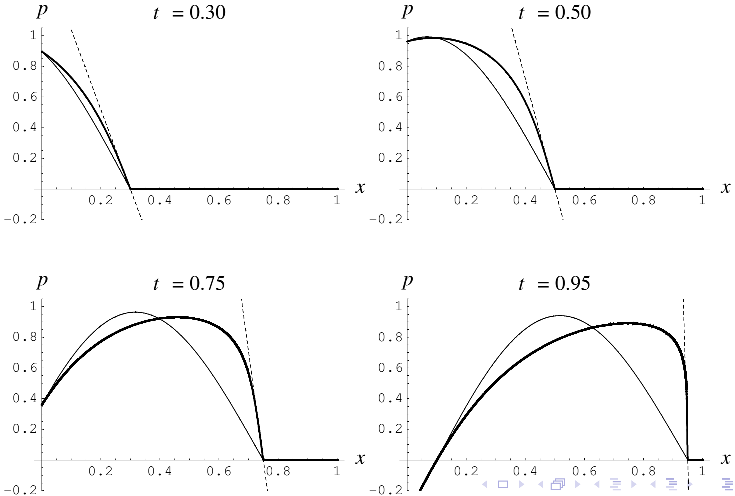
Dimensionless characteristic signal speed: $\approx 1 + \beta \mathrm{Ma} p$,

Since the coefficient of H is "+", this signal is *compressive*; thus, since $\omega > \alpha^\star$ is assumed, a shock will form at $t = t_{\mathrm{sh}}$, where

$$t_{\mathrm{sh}} = \frac{2}{\sigma} \ln\left(\frac{1}{1 - \alpha^\star/\omega}\right), \quad \text{and} \quad \alpha^\star = \frac{\sigma}{2\beta \mathrm{Ma}} \quad (\omega > \alpha^\star). \qquad (28)$$

Numerical Simulation of Poroacoustic Shock Formation

p vs. *x* for air (Jordan, 2005). **Bold curves**: Nonlinear (Ma ≈ 0.238) EoM. Thin curves: Linearized (Ma := 0) EoM. Broken lines: Tangents to bold curves at $x = t$; here, $\omega = 3.7$, $\alpha^{\star} \approx 0.409$, $t_{\mathrm{sh}} \approx 1.0$.



PART V: Uni-Directional Models & Further Reading

Uni-Directional, Weakly-Nonlinear Acoustic Models

Under the additional assumption of uni-directional propagation, the 1D versions of our finite-amplitude acoustic models can be reduced to the following (dimensionless) PDEs:

- The Riemann equation (lossless gases)

$$u_t + (1 + \mathrm{Ma}\beta u)u_x = 0 \qquad \text{(Crighton, 1998)},$$

- Burgers' equation (thermoviscous gases)

$$u_t + (1 + \mathrm{Ma}\beta u)u_x = \tfrac{1}{2}\delta u_{xx} \qquad \text{(Lighthill, 1956)},$$

- The damped Riemann equation (poroacoustic propagation)

$$u_t + (1 + \mathrm{Ma}\beta u)u_x + \tfrac{1}{2}\sigma u = 0 \qquad \text{(Jordan, 2020)},$$

where $0 < \mathrm{Ma} \ll 1$, and in each case propagation to the *right* has been assumed.

Further Reading (+ Refs. cited therein)

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